# On the existence of viscosity solutions to nonlinear problems involving an integro-differential operator(Evolution Equations and Applications to Nonlinear Problems)



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# On the existence of viscosity solutions to nonlinear problems involving an integro-differential operator

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## 1. Introduction

This is a part of the joint work [11] with Suzanne M. Lenhart at University of Tennessee, Knoxville.

In this note we consider the existence of viscosity solutions for an obstacle problem involving an integro-differential operator associated with piecewise-deterministic processes.

Let

$$Lu(x) = -g(x) \cdot \nabla u(x) + \alpha(x)u(x) - \lambda(x)\int_{\Omega} \big(u(y) - u(x)\big)Q(dy, x),$$

where $\cdot$ is the inner product in $\boldsymbol{R}^n$, $\nabla u$ is the gradient vector of $u$ and $Q(\cdot, x)$ is a probability measure.

We consider the following obstacle problem:

$$\min\{Lu - f,\ u - \psi\} = 0 \quad \text{in } \Omega, \tag{1.1}$$

with the boundary condition

$$u(x) = \int_{\Omega} u(z)Q(dz, x) \quad \text{on } \partial\Omega. \tag{1.2}$$

The operator $L$ arises as a generalized infinitesimal generator of a piecewise-deterministic (PD in short) process. These PD processes have deterministic dynamics $g$ between random jumps. The jump distribution is represented by transition probability measure $Q(\cdot, x)$. See Davis [4] for the detail of PD processes.

In the case that $L$ is an infinitesimal generator of a diffusion process, it is well known that the unilateral obstacle problem (1.1) with the Dirichlet boundary condition arises as a dynamic programming equation associated with an appropriate optimal control problem (see Bensoussan and Lions [1]).

The equation (1.1) is also the dynamic programming equation associated with an optimal control problem in which the underlying process is a PD process.

In the case that the domain $\Omega$ is a bounded domain in $\boldsymbol{R}^n$, the PD process jumps back into the interior upon hitting the boundary which leads to the boundary condition (1.2) (see Davis [4]).

The obstacle problem (1.1), (1.2) is first treated by Lenhart and Liao [9], [10] by using singular perturbation method. After introduction of the notion of viscosity solution by Crandall and Lions [2], Lenhart [8] has proved the existence and uniqueness of viscosity solution for a system of obstacle problems.

In these articles, it is commonly assumed that

$$\alpha(x) \geq \alpha_0 > 0 \quad \textit{for sufficiently large } \alpha_0.$$

The perpose of this note is to eliminate the condition of largeness for the zero-th order term by using Perron's method which is introduced by Ishii [6].

In section 2, we state the notion of viscosity solutions and assumptions. We also give a brief review of Perron's method. In section 3, we shall explain how to apply the Perron's method to get a viscosity solution of (1.1) satisfying the boundary condition (1.2). To show the existence of super- and subsolution, which are needed to apply Perron's method, we consider also a linear first order PDE with the boundary condition (1.2). Our main result is Theorem 3.3.

## 2. Assumptions and Perron's method

Let

$$Lu(x) = -g(x) \cdot \nabla u(x) + \alpha(x)u(x) - \lambda(x) \int_\Omega (u(y) - u(x))Q(dy, x), \tag{2.1}$$

where $\cdot$ is the usual inner product in $\boldsymbol{R}^n$, $\nabla u$ is the gradient vector of $u$ and $Q(\cdot, x)$ is a probability measure.

We consider the following obstacle problem.

$$\min\{Lu - f,\, u - \psi\} = 0 \qquad \text{in } \Omega, \tag{2.2}$$

$$u(x) = \int_\Omega u(y)Q(dy, x) \qquad \text{on } \partial\Omega \tag{2.3}$$

We assume the following conditions.

(H.1) $\Omega$ is a bounded domain in $\boldsymbol{R}^n$ with smooth boundary $\partial\Omega$.

(H.2) $g(x) : \Omega \to \boldsymbol{R}^n$ is Lipschitz continuous, $\alpha(x), \lambda(x) : \overline{\Omega} \to \boldsymbol{R}$ are continuous.

(H.3) There exists $\alpha_0 > 0$ such that $\alpha(x) \geq \alpha_0$ for $x \in \overline{\Omega}$.

(H.4) $\lambda(x) > 0$ for $x \in \Omega$.

(H.5) $Q(\cdot, x)$ satisfies:

(i) $Q(\cdot, x)$ is a probability measure on $\Omega$ for $x \in \overline{\Omega}$ such that

$$\left| \int_\Omega v(y)Q(dy, x) \right| \leq C\|v\|_{L^1(\Omega)} \qquad \text{for all } v \in L^1(\Omega).$$

(ii) The function

$$x \to \int_\Omega v(y)Q(dy, x),$$

is continuous with respect to $x \in \overline{\Omega}$, uniformly on $v \in L^\infty(\Omega)$.

(H.6) $g(x) \cdot \eta(x) > 0$ for $x \in \partial\Omega$, where $\eta(x)$ is the outward unit normal at $x \in \partial\Omega$.

(H.7) $f, \psi$ are continuous on $\overline{\Omega}$.

We denote that

$$F(x, u, p, r) = \min\{-g(x) \cdot p + (\alpha(x) + \lambda(x))u - \lambda(x)r - f(x),\, u - \psi(x)\}.$$

for $x \in \Omega, u \in \boldsymbol{R}, p \in \boldsymbol{R}^n, r \in \boldsymbol{R}$. Notice that if we fix $v \in L^\infty(\Omega)$, then the equation

$$F\left(x, u(x), \nabla u(x), \int_\Omega v(y)Q(dy,x)\right) = 0 \qquad \text{in } \Omega$$

is an obstacle problem with a first order Hamiltonian.

We give some notation necessary to state the definition of viscosity solution. For bounded functions, we set

$$u^*(x) = \lim_{r\to 0} \sup\{u(y) \,|\, |x-y| < r\} \qquad \text{upper semi-continuous envelope of } u$$

and

$$u_*(x) = \lim_{r\to 0} \inf\{u(y) \,|\, |x-y| < r\} \qquad \text{lower semi-continuous envelope of } u.$$

Now we state the definition of viscosity solutions.

**Definition.** Let $u$ be a bounded measurable function.

(i) $u$ is a viscosity subsolution of (2.2) if

$$F\left(x, u^*(x), \nabla\phi(x), \int_\Omega u^*(y)Q(dy,x)\right) \le 0$$

wherever $u^* - \phi$ attains its maximum for $\phi \in C^1(\Omega)$.

(ii) $u$ is a viscosity supersolution of (2.2) if

$$F\left(x, u_*(x), \nabla\phi(x), \int_\Omega u_*(y)Q(dy,x)\right) \ge 0$$

wherever $u_* - \phi$ attains its minimum for $\phi \in C^1(\Omega)$.

(iii) $u$ is a viscosity solution if $u$ is a viscosity sub- and supersolution.

In the following, "(sub/super) solution" means "viscosity (sub/super) solution".

Assume that there exists a supersolution $W$ of (2.2) such that

$$W(x) \ge \int_\Omega W(y)Q(dy,x) \qquad \text{on } \partial\Omega. \tag{2.4}$$

Define

$$\mathcal{S} = \{v|\, v \text{ is a subsolution of (2.2) such that}$$
$$v \le W \quad \text{in } \Omega \text{ and}$$
$$v(x) \le \int_\Omega v(y)Q(dy,x) \quad \text{on } \partial\Omega\}.$$

We put

$$u_0(x) = \sup\{v(x)|\, v \in \mathcal{S}\}.$$

Perron's method consists of the following two propositions:

**Proposition 2.1.** *Assume that $\mathcal{S}$ is not empty, then $u_0 \in \mathcal{S}$.*

**Proposition 2.2.** *Assume $\mathcal{S} \neq \emptyset$. If $v \in \mathcal{S}$ is not a supersolution, then there exists $w \in \mathcal{S}$ such that $v(y) < w(y)$ at some $y \in \Omega$.*

These two Propositions can be proved by the same idea of Ishii [6]. So we omit the proofs. See [11] for the detail.

Note that $u_0$ is a viscosity solution of (2.2).

## 3. Main existence result

First we assume that there exists a supersolution $W$ of (2.2) satisfying (2.4). By Perron's method, there exists a solution $u_0$. Note that $u_0$ satisfies the boundary inequality

$$u_0(x) \le \int_\Omega u_0(y)Q(dy,x) \qquad \text{on } \partial\Omega.$$

**Theorem 3.1.** *Assume* (H.1)–(H.7). *Suppose that there exists a supersolution $W$ of* (2.2) *satisfying* (2.4), *and a solution $u_1$ of*

$$F\left(x, u_1, \nabla u_1, \int_\Omega u_0(y)Q(dy,x)\right) = 0 \qquad \text{in } \Omega \tag{3.1}$$

*satisfying the Dirichlet boundary condition*

$$u_1(x) = \int_\Omega u_0(y)Q(dy,x) \qquad \text{on } \partial\Omega. \tag{3.2}$$

*If $u_1 \leq W$, then $u_0$ is a solution of* (2.2) *satisfying the boundary condition* (2.3).

*Proof.* We calim $u_1 \in \mathcal{S}$. Let $\phi \in C^1$ such that $u_1^* - \phi$ attains its maximum at $y_0$, then

$$F\left(y_0, u_1^*(y_0), \nabla\phi(y_0), \int_\Omega u_0(y)Q(dy,y_0)\right) \leq 0.$$

Note that the comparison principle for two viscosity solutions holds for the equation of a first order Hamiltonian $F(x,u,\nabla u,u_0)$. Since $u_0$ is also a subsolution of (3.1), we have $u_0 \leq u_1$ in $\Omega$. Using $u_0 \leq u_1$ and the monotonicity of $F$ with respect to the argument $u$, we have

$$F\left(y_0, u_1^*(y_0), \nabla\phi(y_0), \int_\Omega u_1(y)Q(dy,y_0)\right) \leq 0.$$

Also we have

$$u_1(x) = \int_\Omega u_0(y)Q(dy,x) \leq \int_\Omega u_1(y)Q(dy,x) \qquad \text{on } \partial\Omega.$$

Hence, we have the claim. By the definition of $u_0$ and $u_0 \leq u_1$, we have $u_0 \equiv u_1$ in $\overline{\Omega}$. This completes the proof.

To assure the assumptions of Theorem 3.1, we consider the equation

$$Lu(x) = f(x) \qquad \text{in } \Omega \tag{3.3}$$

$$u(x) = \int_\Omega u(y)Q(dy,x) \qquad \text{on } \partial\Omega. \tag{3.4}$$

**Theorem 3.2.** *Assume* (H.1)–(H.7). *Then there exists a unique solution of the equation* (3.3) *satisfying the boundary condition* (3.4).

*Proof.* First we note that

$$w(x) = -\frac{\|f\|_\infty}{\alpha_0} \quad \text{is a subsolution,}$$

and

$$W(x) = \frac{\|f\|_\infty}{\alpha_0} \quad \text{is a supersolution.}$$

of (3.3) satisfying (3.4).

Applying Perron's method, we have that there exists a solution $u_0$ of (3.3) satisfying the boundary inequality

$$u_0(x) \leq \int_\Omega u_0(y)Q(dy, x) \qquad \text{on } \partial\Omega.$$

Next we consider the equation

$$-g \cdot \nabla u_1 + (\alpha + \lambda)u_1 - \lambda \int_\Omega u_0(y)Q(dy, x) = f \qquad \text{in } \Omega \tag{3.5}$$

with the Dirichlet boundary condition

$$u_1(x) = \int_\Omega u_0(y)Q(dy, x) \qquad \text{on } \partial\Omega. \tag{3.6}$$

The comparison principle for this equation is well known [2,3]. By (H.6) and the method of [12], we can prove the existence of sub- and supersolutions. Then there exists a continuous solution $u_1$ of the equation (3.5) with (3.6). We can apply the same argument in the proof of Theorem 3.1 to yield that $u_1 \equiv u_0$. The uniqueness follows from Lenhart [8]. The proof is complete.

Now we can prove the main result.

**Theorem 3.3.** *Assume* (H.1)–(H.7). *Then there exists a unique solution of the obstacle problem* (2.2) *satisfying the boundary condition* (2.3).

*Proof.* It is sufficient to check the hypothesis of Theorem 3.1. To do so, we consider the obstacle problem (3.1) with (3.2).

Using the boundary inequality of $u_0$ and $u_0 \geq \psi$ in $\Omega$, the compatibility condition

$$\psi(x) \leq \int_\Omega u_0(y)Q(dy,x) \qquad \text{on } \partial\Omega$$

is satisfied.

First assume

$$h(x) = \int_\Omega u_0(y)Q(dy,x) \in C^1(\Omega) \cap C(\overline{\Omega}) \tag{3.7}$$

and

$$h(x) = \int_\Omega u_0(y)Q(dy,x) > \psi(x) \qquad \text{on } \partial\Omega \tag{3.8}$$

In this case, problem (3.1) with (3.2) is equivalent to

$$\min\{-g \cdot \nabla w_1 + (\alpha+\lambda)w_1 - f,\ w_1 - \psi\} = 0 \qquad \text{in } \Omega \tag{3.9}$$

$$w_1(x) = 0 \qquad \text{on } \partial\Omega \tag{3.10}$$

where $f, \psi$ satisfy the same properties as $f, \psi$ in (3.1) and $\psi < 0$ on $\partial\Omega$. We show the existence of a solution to (3.9) with (3.10) by Perron's method. Indeed, the solution of the linear equation

$$-g \cdot \nabla w + (\alpha+\lambda)w = f \qquad \text{in } \Omega,$$

$$w = 0 \qquad \text{on } \partial\Omega$$

is a subsolution of (3.9) with (3.10).

To construct a supersolution, we follow a barrier construction argument from Oleinik and Radkevic [12] as in Ishii and Koike [7]. Since $\psi < 0$ on $\partial\Omega$, there exists a local barrier, $\psi_z$ in $C(\Omega \cap V_z) \cap C^2(\Omega \cap V_z)$ where $z \in \partial\Omega$, $V_z$ is a sufficiently small neighborhood of $z$ satisfying

$$\psi_z(z) = 0, \quad \psi_z \geq 0 \quad \text{on } \overline{\Omega \cap V_z},$$

$$\psi_z \geq \|f\|_\infty/\alpha_0 \quad \text{on } \overline{\Omega} \cap \partial V_z,$$

$$-g \cdot \nabla\psi_z + (\alpha+\lambda)\psi_z \geq f \quad \text{in } \Omega \cap V_z, \text{ and}$$

$$\psi_z \geq \psi \quad \text{in } \Omega \cap V_z.$$

Define

$$\hat{\psi}_z(z) = \begin{cases} \max\{\psi_z(x),\ \max\{\|f\|_\infty/\alpha_0,\ \|\psi\|_\infty\}\} & \text{in } \Omega \cap V_z, \\ \max\{\|f\|_\infty/\alpha_0,\ \|\psi\|_\infty\} & \text{otherwise,} \end{cases}$$

and

$$\hat{\psi}(x) = \inf\{\hat{\psi}_z(x) |\ z \in \partial\Omega\}.$$

Then $\hat{\psi}$ is a supersolution. This implies that there exists a continuous solution of (3.1) with (3.2).

For general continuous boundary value $h$, which is not necessarily satisfy (3.7) and (3.8), we choose an approximating sequence $\{h_n\}$ such that $h_n \in C(\Omega) \cap C^1(\Omega)$, $h_n > \psi$ on $\partial\Omega$ and $h_n \to h$ uniformly in $\overline{\Omega}$. Let $u_n$ be a solution of (3.1) with (3.2) associated with boundary value $h_n$. By standard comparison argument, we have

$$\sup_{\Omega} |u_n(x) - u_m(x)| \leq \sup_{\partial\Omega} |h_n(x) - h_m(x)|.$$

Hence $\{u_n\}$ converges to some $u \in C(\overline{\Omega})$ and by stability of viscosity solutions, we have that $u$ is a solution of (3.1) with (3.2).

By the comparison result for obstacle problems, we have $u_1 \leq W$. Hence by Theorem 3.1, $u_0$ satisfies the boundary condition (3.2).

Since the uniqueness follows from the argument in Lenhart [10], the proof is completed.

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